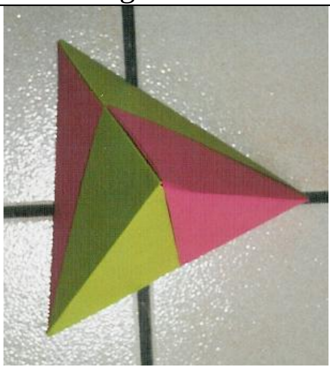
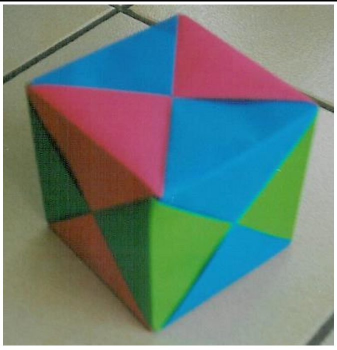
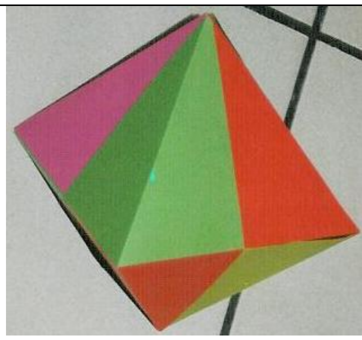
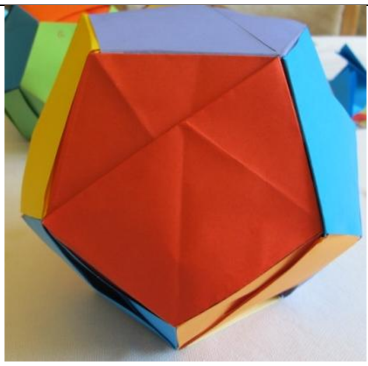
Regular stellated Polyhedra or Kepler-Poinsot Polyhedra by Origami.

Marcel Morales
Institut Fourier, Université de Grenoble I, Grenoble France.
UMR 5582, B.P.74,38402 Saint-Martin D'H\`eres
ESPE Université de Lyon 1, Lyon, France.
morales@ujf-grenoble.fr

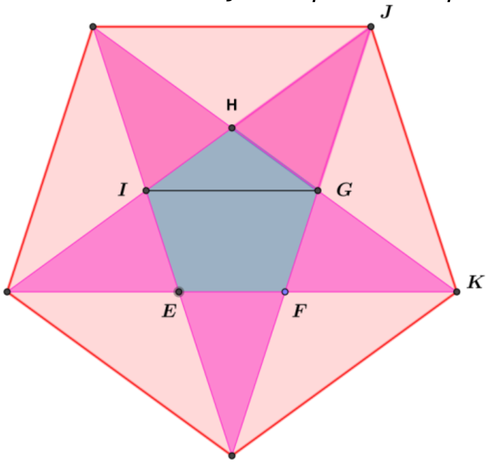
Two thousand years ago, Platon described the five convex regular Polyhedra, the Tetrahedron, the Cube, the Octahedron, the Dodecahedron and the Icosahedron. All of them except the Dodecahedron can be constructed by folding paper, they are part of the traditional Japanese art of Origami.

 **Tetrahedron**	 **Cube**	 **Octahedron**

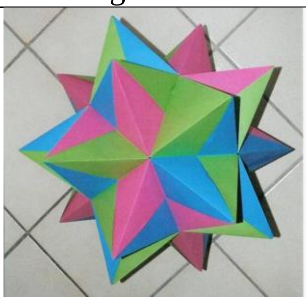
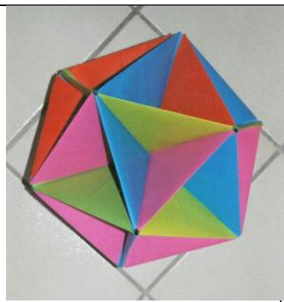
 **Dodecahedron**	 **Icosahedron**

The stellated regular Polyhedra were discovered by Kepler and Poinsot.
Johannes Kepler, discovers in 1619 two non convex regular Polyhedra:
the Small Stellated Dodecahedra and the Big Stellated Dodecahedra (Kepler's stellation).
Two centuries later, in 1809 Louis Poinsot discovers, two other non convex regular Polyhedra: the Great Dodecahedra and the Great Icosahedra.
They are called regular for several reasons, the first one by tradition and the second one because every visible face (convex or Stellated regular pentagon, or equilateral triangle) is in a plane.
The Small Stellated Dodecahedra has 12 faces (star pentagons called pentagrams), 30 edges, 20 vertices.
The Great Stellated dodecahedron has 12 faces (star pentagons called pentagrams), 30 edges, 20 vertices.

The Great dodecahedron has 12 faces (pentagons), 30 edges, 20 vertices.
The Great icosahedron has 20 faces (equilateral triangles), 30 edges, 12 vertices.
They can be constructed by patterns thanks to the work of H.S.M.Coxeter (e.g. his nice book *Regular polytopes*) and the book *Polyhedron Models*, by Magnus Wenninger.
The aim of this work is to present a simple method to construct them by Origami. Our models are exact and are they are produced for the first time in this work and in a book for children that will appears very soon. I have developed these models in order to teach the Geometry in the high schools. My motivation comes since my teaching duty consist to form the future high schools teachers. I have presented my Origami method to teach Geometry in many Universities of 10 countries in four of the five continents. This method was experimented in high school classes, pupils like a lot to construct the Polyhedra and can be able to explain many non elementary properties of Geometry only by intuition and by handling the model.
Since we want to produce exact models of the regular Polyhedra by Origami we need a famous rectangle, called golden rectangle of size 1 by φ, where φ is the golden number, approximately we have that φ=1.6180339887. The golden rectangle is famous because it is used in arts. Any rectangle proportional to the golden rectangle is also called golden rectangle. The golden number appears naturally in the regular pentagon. The quotient of the diagonal GI by the side GH is the golden number. The quotient of the side of the big pentagon JK by GK is the golden number. Also GI=GJ. $JK = \varphi^2 = 1 + \varphi$.



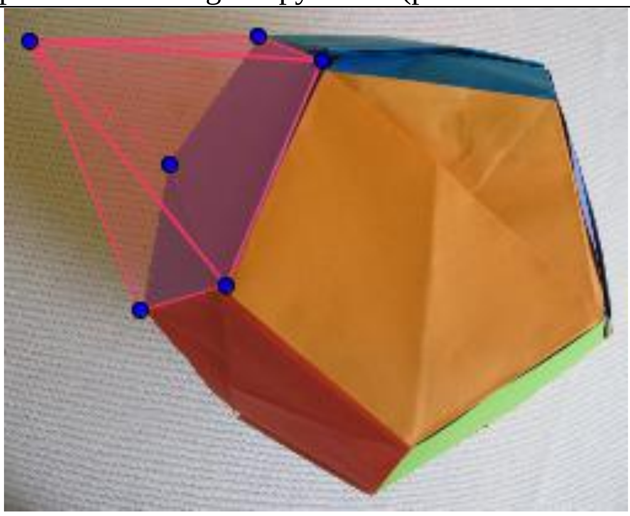
The classification of convex regular Polyhedra is based only in their group of rotations. This is not the case for Stellated Polyhedra. There are several Stellated Polyhedra with the same group of rotations, for this reason we can find many nice Stellated Polyhedra that could be called regular.

			
The Delta small star Dodecahedron	**The Delta great star Dodecahedron**	**The Delta great Dodecahedron**	**The Delta great Icosahedron**

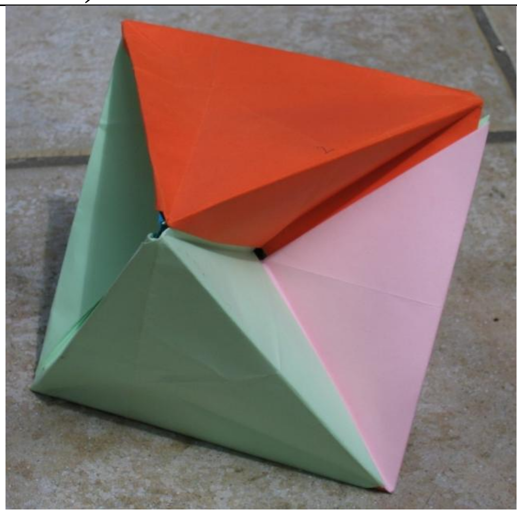
For the construction of these Polyhedra please visit my web page:
https://www-fourier.ujf-grenoble.fr/~morales/regular-polyhedra-english.html

Star Polyhedra. Stellation and Excavation.

Roughly speaking to produce a star polyhedron from another polyhedron, consist to take a plane face of the polyhedron and put outside a regular pyramid (process called stellation) or put it inside a regular pyramid (process called excavation).



Stellation



Excavation

Now I will speak more precisely about the regular stellations of the Dodecahedron and the Icosahedron which give very nice Polyhedra, among them the so called regular stellated Polyhedra or Kepler-Poinsot Polyhedra, and the third stellation of the Icosahedron.

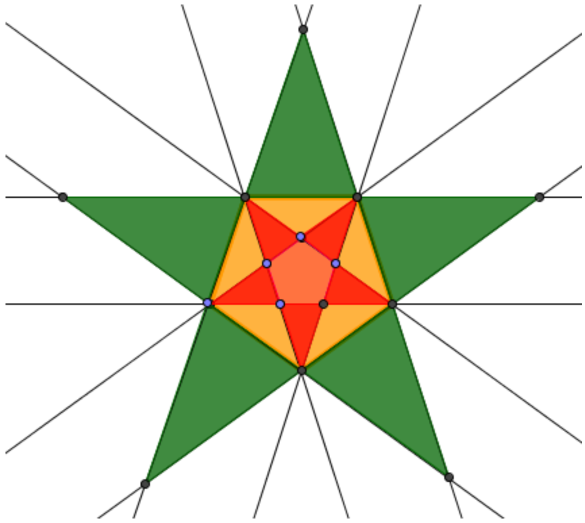
A face of a convex polyhedron is the figure that we can see in a polyhedron which is contained in a plane. In the case of the Dodecahedron a face is a regular pentagon, in the case of the Icosahedron a face is an equilateral triangle.

Let consider all the planes that contain faces of the polyhedron, we call it a face-plane of the polyhedron. Recall that two planes non parallel cut into a line.

The mathematician Coxeter fixed one of the face-planes of a polyhedron and draws in this plane all the lines resulting from cutting this plane with all other non parallel face-plane of the polyhedron.

The Dodecahedron has only three stellations.

For the Dodecahedron we have 12 face-planes, we fix one of them, then we have 10 lines because they are 10 face-plane cutting this fixed face-plane. The 11^{th} face-plane is parallel to the fixed one. We get the following picture, the 10 lines and their intersection



From this picture we get all the stellations of the Dodecahedron. We will explain them:

- The small stellated Dodecahedron:

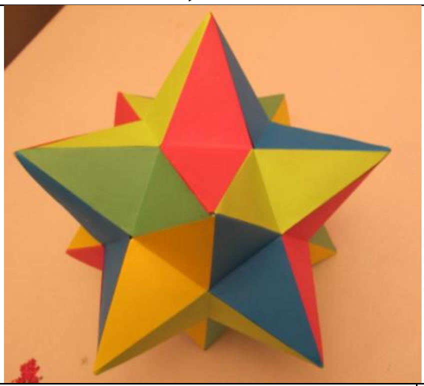
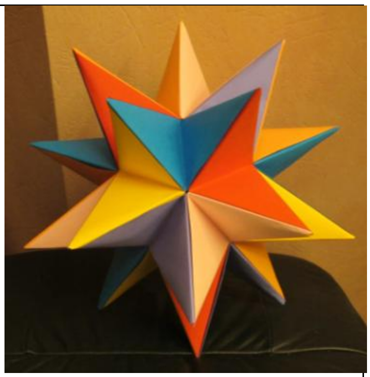
Every visible face of the small stellated Dodecahedron is given by the regions colored in red, as you can see is a star pentagon.

- The great Dodecahedron:

Every visible face of great Dodecahedron is given by the regions colored in yellow, as you can see it consist of 5 triangles in a regular pentagon.

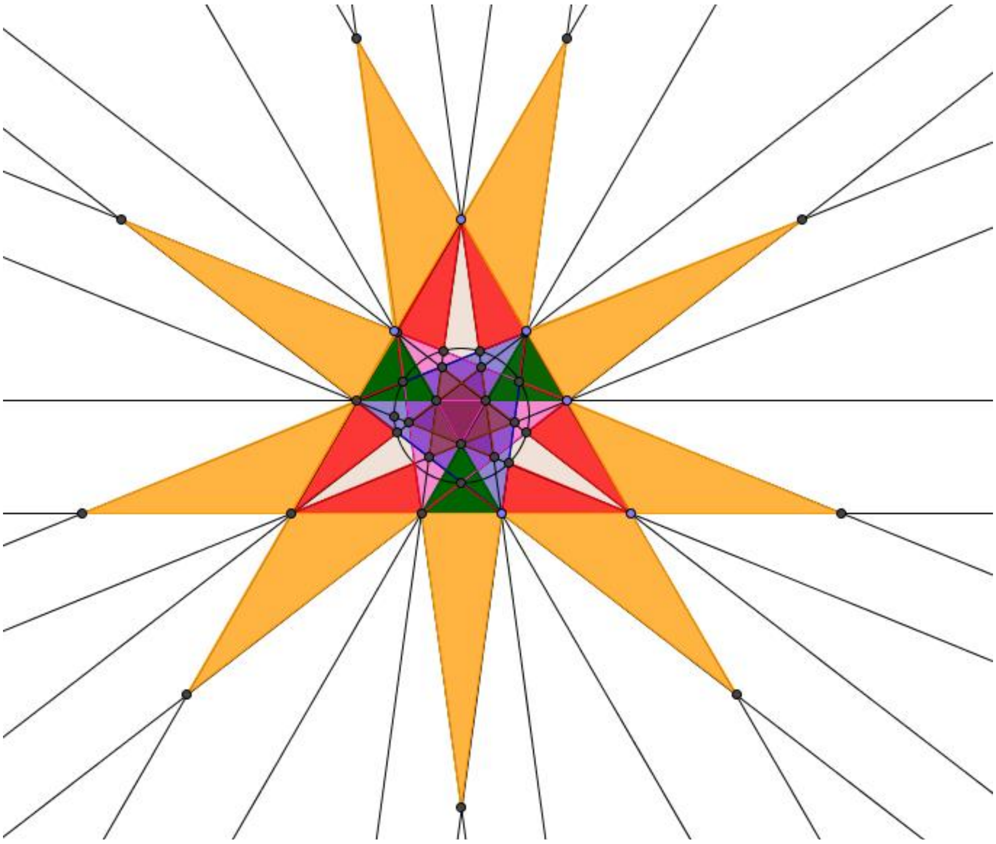
- The Great Stellated Dodecahedron:

  If we take the face-planes of the great icosahedron we get the final stellation of the Dodecahedron, named the Great Stellated Dodecahedron.

		
The small stellated Dodecahedron	The great Dodecahedron	The Great Stellated Dodecahedron

The stellations of the Icosahedron.

For the Icosahedron we have 20 face-planes, we fix one of them then we have 18 lines because they are 18 face-plane cutting this fixed face-plane. The 19th face-plane is parallel to the fixed one. We get the following picture, the 18 lines and their intersection.



From this picture we get all the stellations of the Icosahedron. We will explain three of them:

- The great (regular) Icosahedron:

Every visible face of the great (regular) Icosahedron is given by the regions colored in red.

- The third stellation of the Icosahedron:

Every visible face of the third stellation of the Icosahedron is given by the regions colored in green.

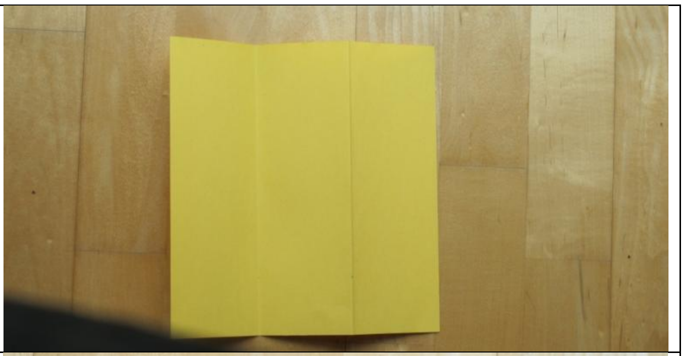
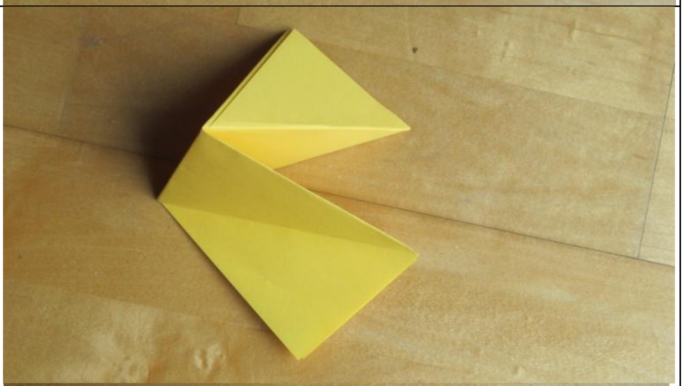
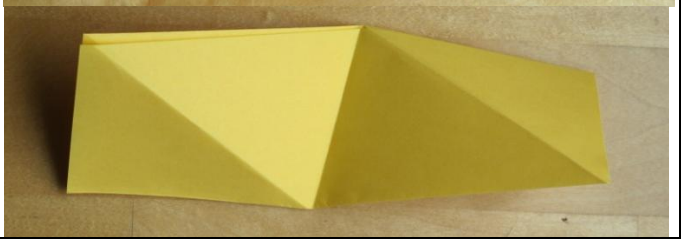
- The final stellation of the Icosahedron :

Every visible face of the final stellation of the Icosahedron is given by the regions colored in yellow.

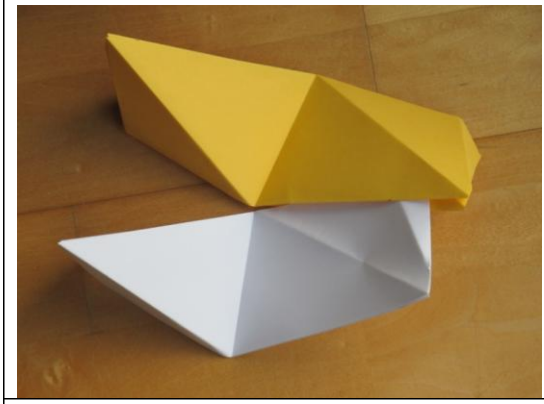
 The great (regular) Icosahedron	 The final stellation of the Icosahedron	 The great star dodecahedron inversed or the third stellation of the icosahedron

Regular Stellated Polyhedra by Origami.

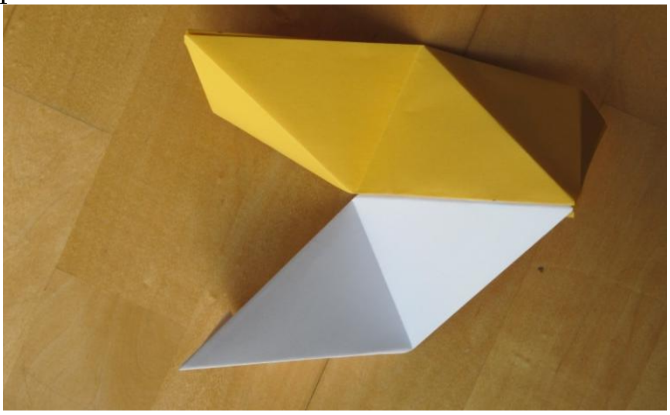
I) Assembly the regular small star dodecahedron

A standard A4 sheet of paper is 210mmX297mm. We will need a rectangular sheet of paper 210mm by 215.4mm. We fold it into three equal parts.	
Each small rectangle has size 70mm X 215.4mm We need a rectangle $\boldsymbol{a}, \boldsymbol{a} \times \sqrt{\mathbf{4}\boldsymbol{\varphi} + \mathbf{3}}$ Note here $\boldsymbol{a} = \mathbf{70}$, $\sqrt{\mathbf{4}\boldsymbol{\varphi} + \mathbf{3}} \sim \mathbf{3{,}07768}$	
We fold the rectangle by overlapping the left down corner over the right top corner. On each part appears two triangles Put on the table and fold them.	
We get the module, it consist of two isosceles triangles which will form the faces of the Polyhedron and two rectangle triangles, these are useful to assembly our Polyhedron. The isosceles triangles are golden triangles, that is the ratio of the two distinct sides is the golden number. **We need 30 modules to produce the Small Stellated Dodecahedron**	 

We must assembly the two modules. Insert the tab of a white module into the slot of the yellow module.



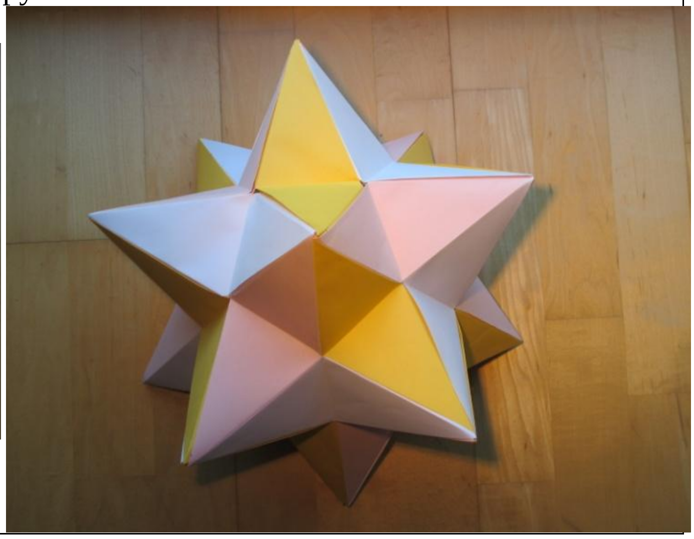
After complete insertion we have the following picture



Continue to insert the tab of one module into the slot of another module. Continue in this way until you have assembled 5 modules. We get the following picture

<u>The Small Stellated Dodecahedron.</u>

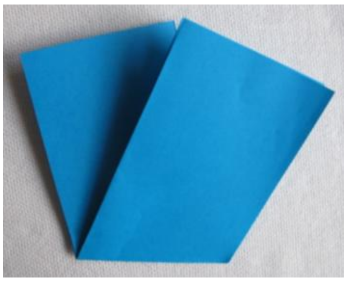
We observe that around each pyramid there are five pyramids.



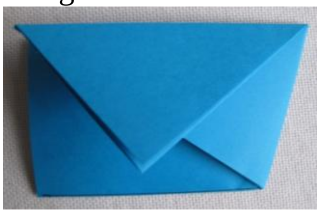
**The Small Stellated Dodecahedron.
is called regular because the star pentagon (here formed by the five triangles colored in red) is in one plane. The Small Stellated Dodecahedron.
is formed by 12 of these star pentagons.**

II) Assembly of the regular Great Dodecahedron

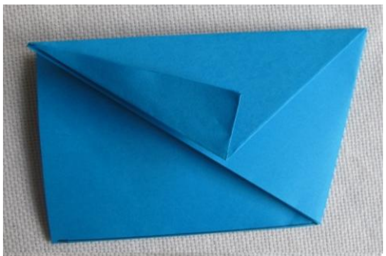
Take a sheet of paper folded 3 times as for The Small Stellated Dodecahedron.
This rectangle has size 70mm X 215.4mm
We fold the rectangle by overlapping the left down corner over the right top corner.

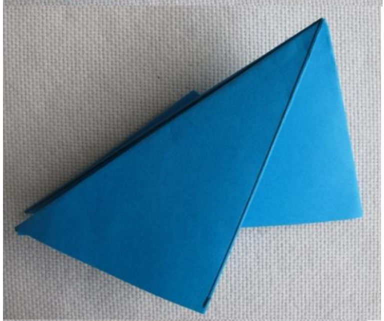


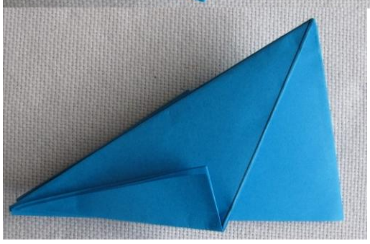
On each part appear two triangles. Fold them.
You can see here the first triangle folded



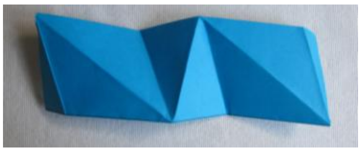
On each part appear two triangles. Fold them.
You can see here the first triangle folded







Cut it as show into two identical pieces. Each piece is a module for the Great Dodecahedron.



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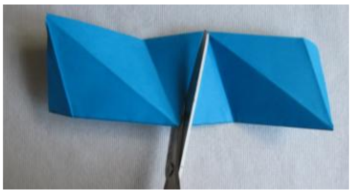
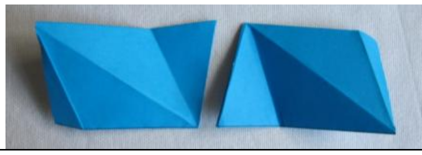


Each piece consist of two isosceles triangles which will form the faces of the Polyhedron and two triangles, these are the tabs to assembly our Polyhedron. The isosceles triangles are golden triangles, that is the ratio of the two distinct sides is the golden number.

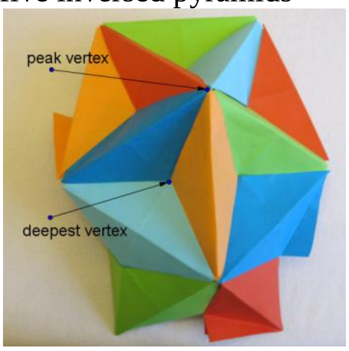


We need 30 modules to produce the Great Dodecahedron. We start with 3 of them. Insert the tab of a red module into the slot of the blue module. The tab of the blue module into the slot of the orange module, and so on.

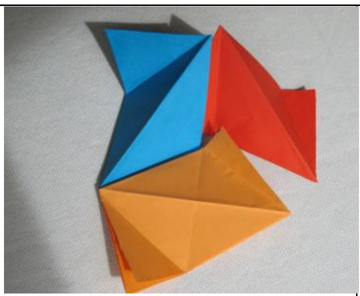
Continue to add modules.

Turn on so that the three pyramids are inversed (excavated)

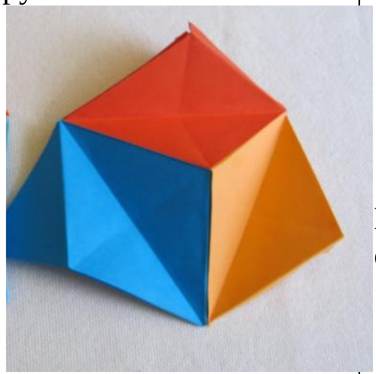
The peak vertices and the deepest vertices. Around each deepest vertex we have three triangles. Around each peak vertex we have five inversed pyramids



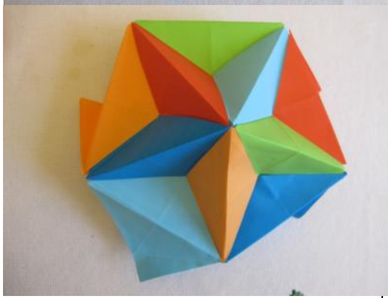
The last module is pink.



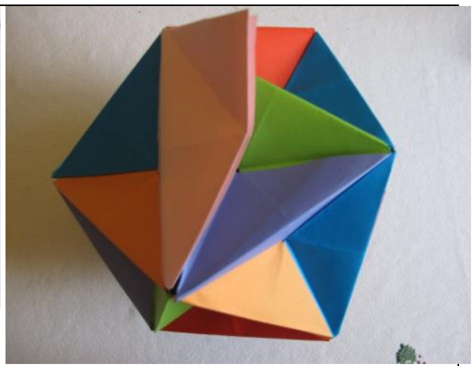
Turn on you can see that you have a triangular pyramid.



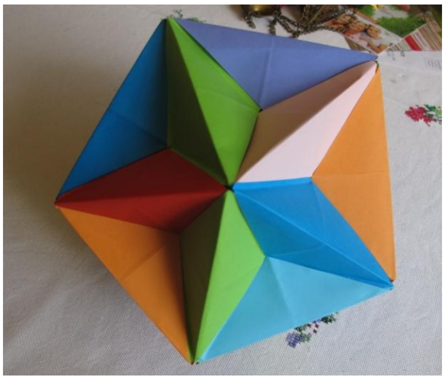
Each free tab or free slot will be used to insert a new module



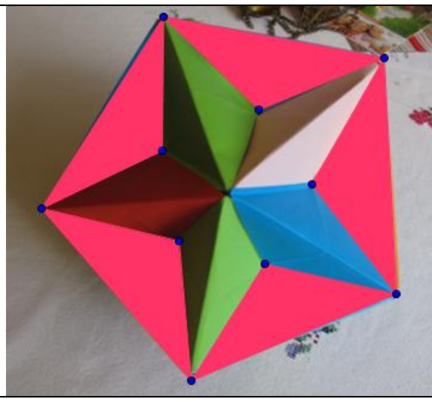
Note that we have two kinds of vertices.



The Great Dodecahedron.



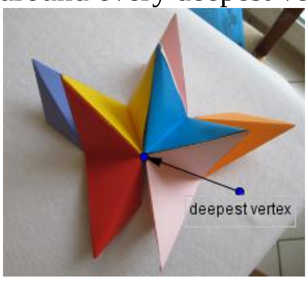
The Great Dodecahedron.
is called regular because the pentagon (formed by 5 triangles here colored in red) is in one plane. The Great Dodecahedron.
is formed by 12 of these pentagons.



III) <u>Assembly of the regular Great Stellated Dodecahedron</u>

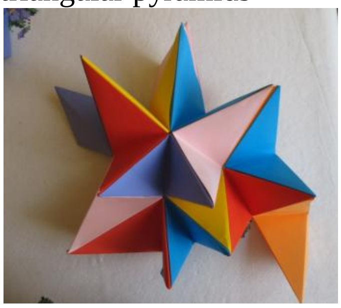
To assembly the Great Stellated Dodecahedron, we need 30 modules identical to those e needed for the Small Stellated Dodecahedron.

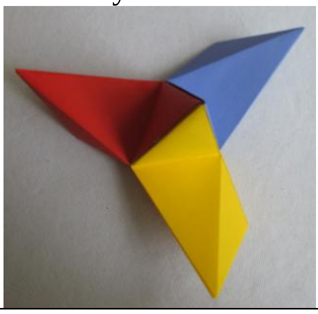
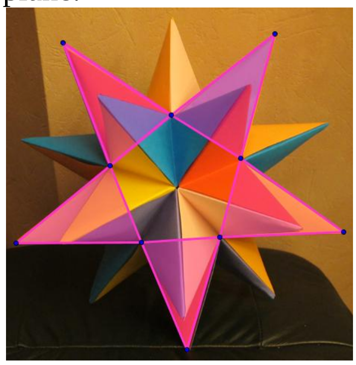
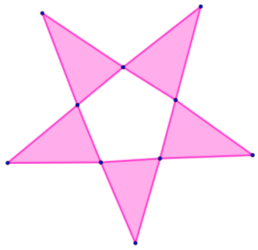
Now we continue to add modules. Remember that must have five triangular pyramids around every deepest vertex.



We have five triangular

Keep in mind that around any deepest point you should have exactly five triangular pyramids

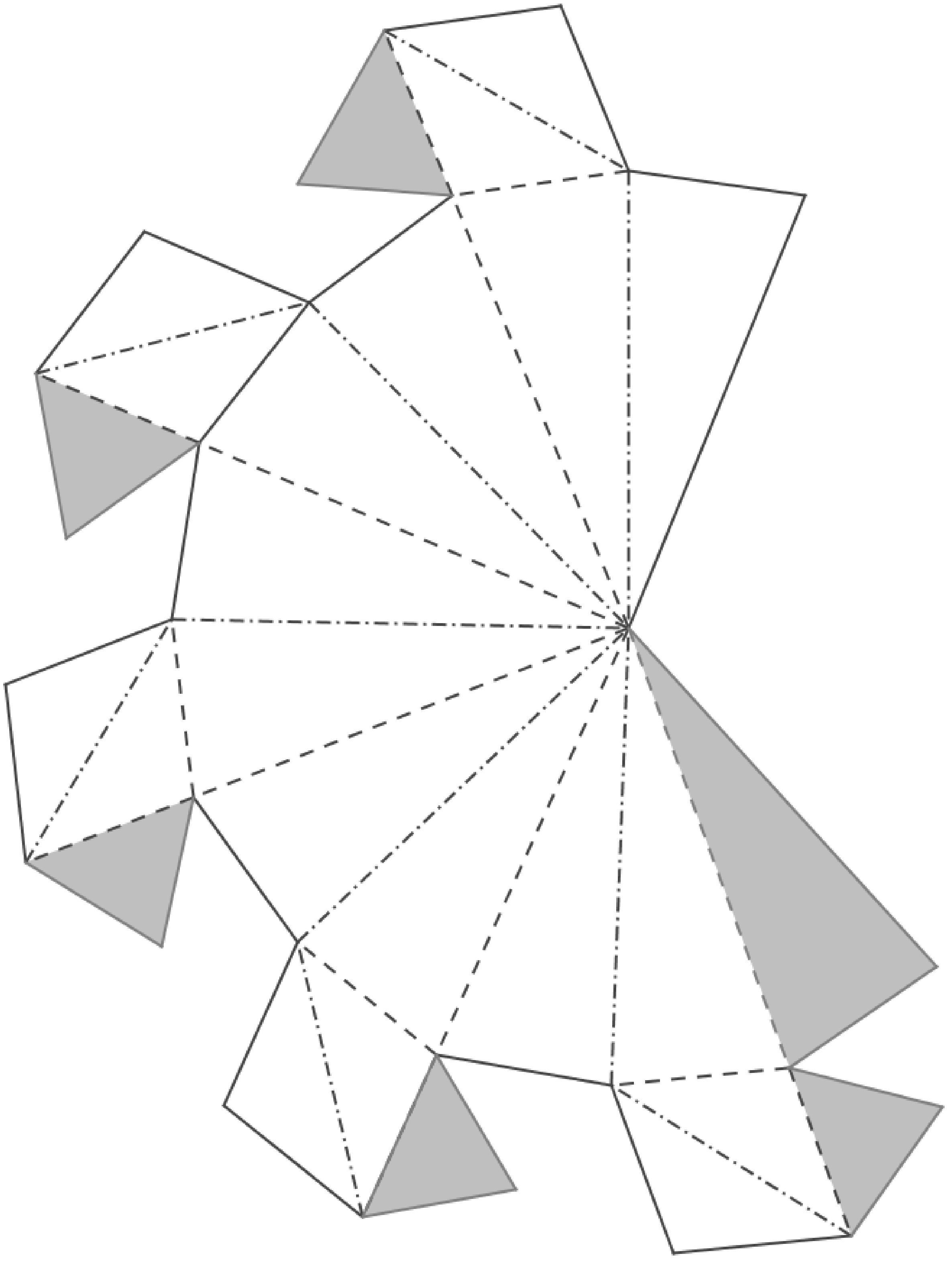


 Assembly three modules. 	pyramids around a deepest vertex.  Continue to join new modules.	We are almost at the last step. 
The Great Stellated Dodecahedron. 	We remark that the star polygon colored in red is in a plane. 	 For this reason the Great Stellated Dodecahedron **is called regular.** The Great Stellated Dodecahedron is formed by 12 of these star pentagons.

The Dodecahedron and its three stellations.

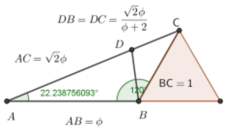
IV) Assembly of the regular Great Stellated Icosahedron

Here is our module to build the Great Stellated Icosahedron. We advise to use paper 160 g per 1square meter. To assembly the Great Stellated Icosahedron, we

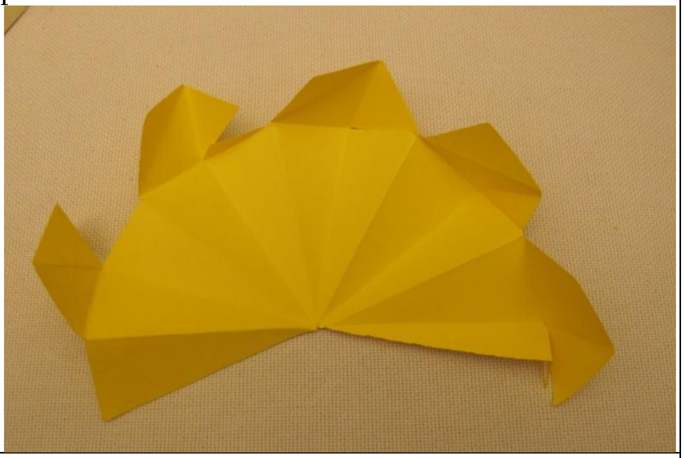


need 12 modules identical to the above one. You have to fold all segments. All the gray triangles will be useful to assembly and will not appear later.

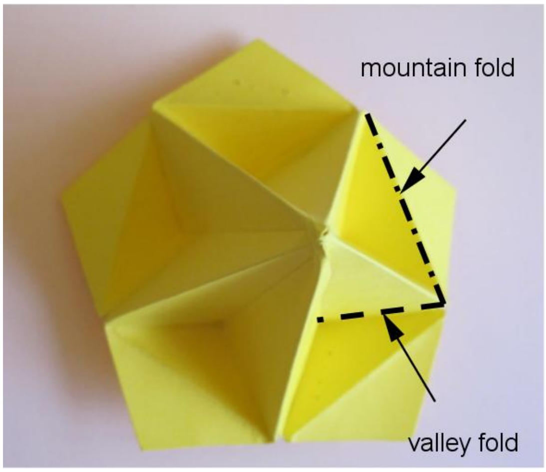
We give the precise size of the basic triangle which appears in our module to build the Great Stellated Icosahedron. Of course our basic triangle is proportional to this one.



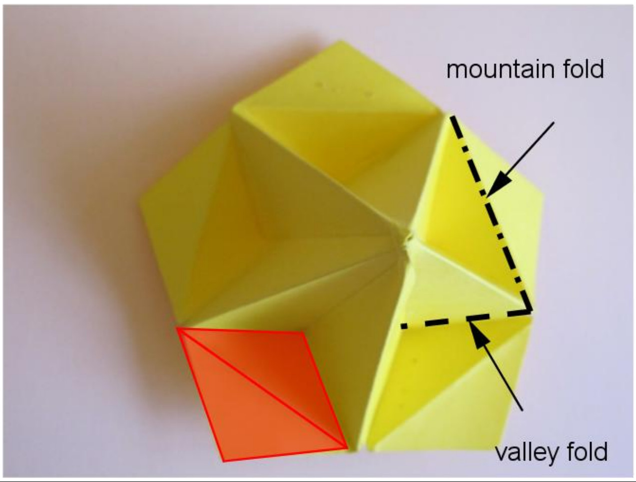
After fold we get the following module.
Recall that we have two kinds of folds: valley fold and mountain fold, as you can see in the next picture.



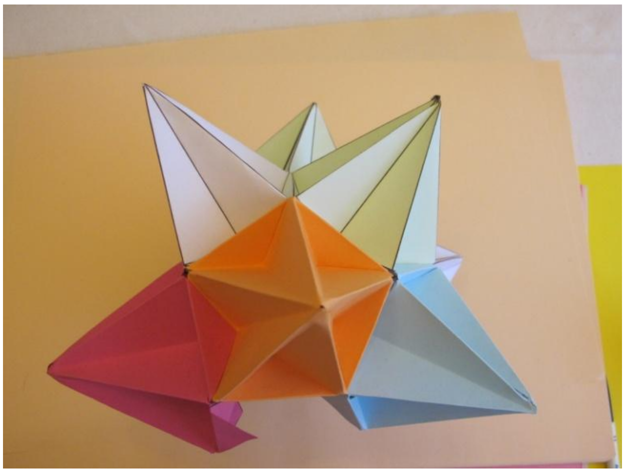
We advise to put a little glue when overlapping the pink triangles. Every module is a pyramid with 5 inversed pyramids.



When you have prepared your 12 modules to assembly them is very easy. You overlap two triangles (colored here in orange) of one module with the corresponding two triangles of a second module.



Here you can see our work after glue together seven modules.

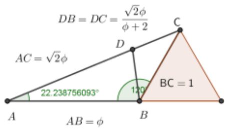


The Great Stellated Icosahedron.

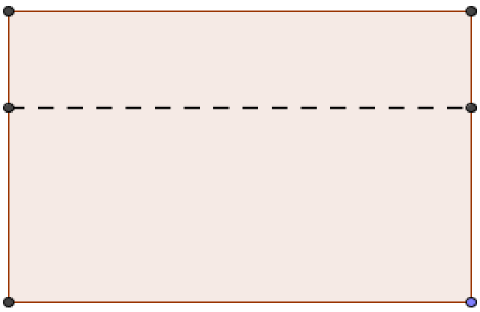
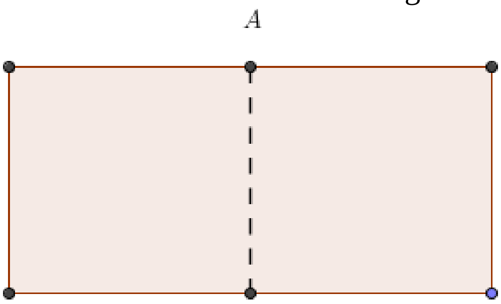
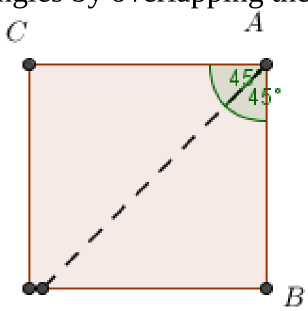
we can remark some triangles in the same plane, colored in red in the picture. For this reason the Great Stellated Icosahedron is called regular. The Great Stellated Icosahedron is formed by 20 planar triangles		

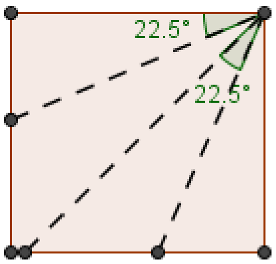
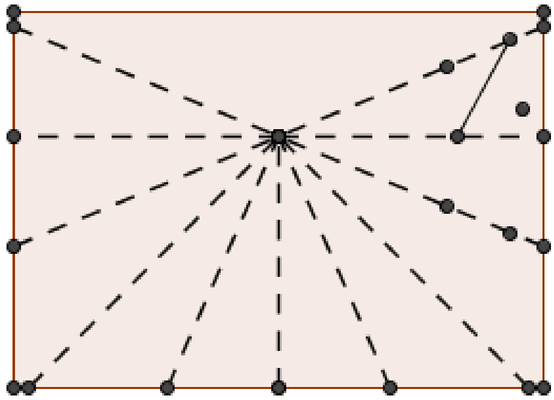
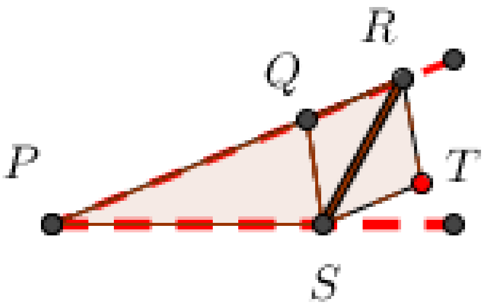
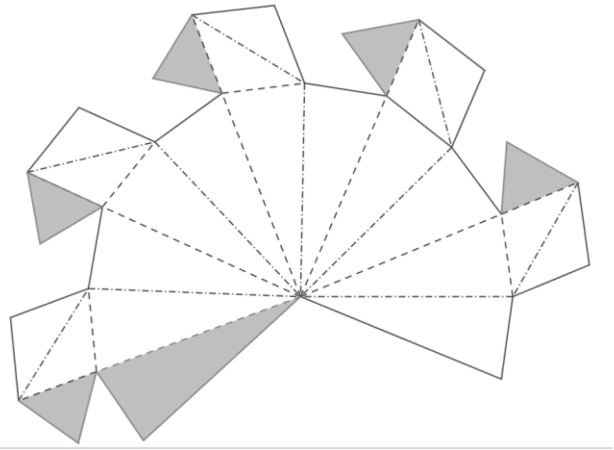
<u>Some remarks on the module needed to build the Great Stellated Icosahedron:</u>

In the exact module need to build the Great Icosahedron, we have an exact triangle.



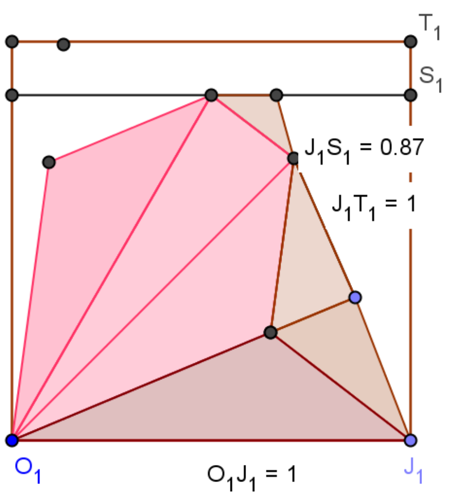
Note that   22. 238756093° is approximately close to 22.25°, and 22.25°=90°/4.
Now we give a method to produce a very good approximation of the exact module by folding paper. In fact   due to the errors produced folding paper the exact module and the approximation will give the same polyhedron:

Take a sheet of paper A4. Draw a line in the third of the height. 	Fold it 
We fold into two identical rectangles 	Now we divide the angle in A into two equal angles by overlapping the segment AB over AC 

Divide again each angle in A into two equal angles by overlapping segments. Fold all lines.  Unfold our sheet of paper. Now we have to size.	We need a golden rectangle. We choose a golden rectangle 6.18cm by 10cm. 
In the basic triangle we have PS=10cm, PQ=10.156cm , PR= 10.91cm, SR=6.18cm SQ=QR=ST=RT= 3,93cm  	 A very good approximation of our module

The final stellation of the Icosahedron.

The method is mixed of Origami and Pattern. We need 60 modules as the one presented here. Each one will form a triangular pyramid. We assembly 5 pyramides together and then the built the final stellation of the Icosahedron. As above the gray triangles will be useful to assembly by using glue and will not appear later.



T_1
S_1
$J_1S_1 = 0.87$
$J_1T_1 = 1$
O_1
$O_1J_1 = 1$
J_1