Plain-woven baskets as hypermaps

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Figure 1 : *Hypermaps are closely associated with ramified coverings of the sphere—what a layman may think of as seamlessly covering other surfaces with copies of the earth's surface. Here is a flat passage realized in knotology weaving, a versatile technique that uses only straight weaving elements.*

Abstract

Baskets that are *plain-woven* (i.e., woven in a strictly over-one-under-one rhythm) embody *alternating links,* and thus can be represented by face 2-colored (a.k.a., *chess-colored*) 4-regular graphs drawn on a closed surface. We may say that plain-woven baskets are represented topologically by chess-colored, 4-regular *maps.* Some sculptors and weavers have advocated practical use of such a representation. The dual of a map contains the same information as the original map—the roles of faces and vertices are merely switched. In particular, a chess-colored, 4-regular map has a dual that is a vertex 2-colored (a.k.a., *bicolored*) quad-faced map. Since every bicolored map represents a *hypermap,* the dual representation of a plain-woven basket is in fact a hypermap. This hypermap representation is recommended by its web of mathematical connections. As one example, we show how a basket woven using Heinz Strobel's knotology technique can model a *Belyi surface.* We demonstrate this visually using Adams's *World in a Square II* conformal projection.

Introduction

As recently as the last decades of the twentieth century, researchers in many fields were working with the same mathematical object, calling it by different names and largely unaware of each other's work. Choosing one of its many guises to serve for a name, we choose to call this mathematical object *hypermaps.* A book by Lando and Zvonkin [1] surveys an amazing breadth of hypermap appearances in many fields of mathematics and in quantum physics. Add basket weaving to that list, as it is easily shown that a known representation of plain-woven baskets has a dual representation that is a hypermap.

Graphs and Hypergraphs

A graph may be thought of as a dots-and-lines drawing, but more abstractly, it is a pairwise relation between elements of a set. For example, consider a set composed of persons. A suitable graph relation could be expressed by the sentence, " _________ and _________ are friends." This is an *undirected* graph relation: any pair of names that make the sentence true can fill the blanks in either order. (We will only be concerned here with undirected graphs.)

Throughout its history, graph theory has been nagged by two questions. Framing those two questions in terms of our example: "Can there be multiple instances of friendship between the same Peter and Paul?" and, "Can Ted be friends with himself?" It has gradually become clear that both questions are best answered in the affirmative—but technical problems arise. In the former case, instances of the relation can no longer be uniquely identified by a two-element set; in the latter case, instances of the relation can no longer be identified with a set at all, since it is a postulate of set theory that an element cannot appear with *multiplicity* other than one.

Saying,"Yes," to yet a third question proved a way out. Framed in terms of our example, the third question is: "Can Peter and Paul and Mary and Peter be friends?" In other words, shall we permit a graph relation to unite any number of set elements having any multiplicity? Answering this in the affirmative gives birth to the concept of a *hypergraph.* Hypergraphs elegantly solve the technical problems mentioned above, but they do this by forcing us back to our first instinct: drawing graphs.

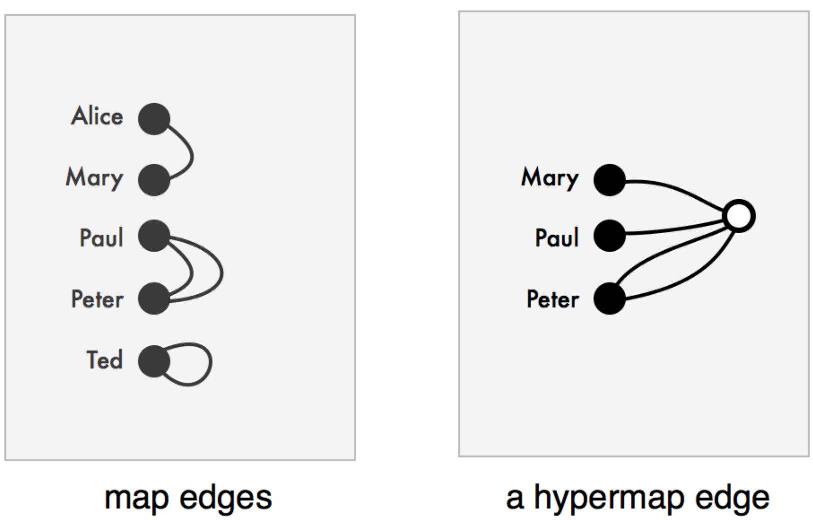

Figure 2 : *An ordinary map edge is incident to two vertices (or one vertex with a multiplicity of two,) a hypermap edge is incident to any number of vertices with any multiplicity.*

Drawing Graphs on Surfaces

Building on the example in the preceding section, suppose we list the names of the persons in the set, and draw a dot beside each name. Then we may make a drawing of the graph by drawing a line between two dots if the corresponding persons are indeed friends. We draw one line for each instance of friendship, drawing as many lines as are called for between the same two dots. If a person is indeed friends with himself, we draw a line that is a *self-loop* from that dot and back—drawing as many self-loops as are called for.

It is not likely that we will find that our set of persons comprises a single network of friends—a graph need not consist of a single connected component—but, anticipating where we intend to go, we will only draw connected graphs. Therefore, at this point in our graph drawing example, we must choose one network of friends (one connected component of the graph,) erasing any dots or lines representing persons or friendships not included in it.

A messy aspect of our graph drawing is that there may be lines crossing. Given some luck, we might be able to reposition dots or reroute lines to avoid line crossings, but a universal solution is to inserting a handle at that location in the surface, making an overpass so that lines can cross over without intersecting.

Having successfully drawn the graph on the surface without crossing lines, there is one more requirement for a proper drawing. We can now define *faces* as the fragments of the surface that would remain if the drawing's lines were to dissolve the surface like acid. In a proper drawing of a graph on a surface, a face must be a simply-connected region. Put another way, a face cannot contain any topological widgets: no holes, no boundaries no handles, no crosscaps. Note that this requirement bars us from drawing graphs on the plane, because any drawing on the plane has an outer face that is not simply connected. A proper graph drawing must be drawn on a closed surface.

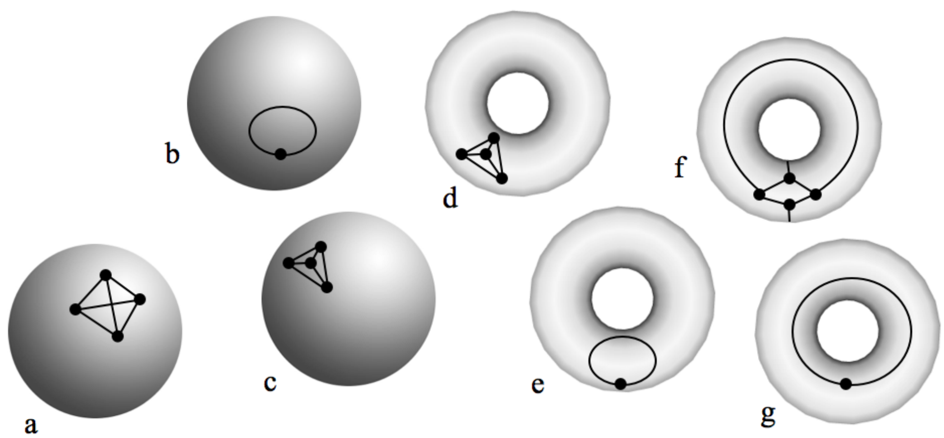

Figure 3 : *Which of these graph drawings are properly drawn on their surfaces? Only b, c, and f.*

Maps

A proper graph drawing, considered up to topological equivalence, is called a *map.* By topological equivalence we mean that in comparing two maps, faces can be stretched or shrunk at will, but their adjacencies cannot be altered.

Every map has a *dual* that can be constructed by placing a new vertex in the center of each face and connecting these new vertices together with edges iff the faces they descend from were adjacent in the original (*primal*) map. The dual switches the roles of vertices and faces, while rotating edges 90°. The dual's dual is the original map. Some maps are described as *self-dual* because the two maps in the pair are topologically indistinguishable; however, when drawn in their constructed relationship on the same surface, it is always possible to distinguish a map from its dual.

So, how might we extend the idea of maps to hypermaps? In particular, how can we draw a graph edge that connects any number of vertices with any multiplicity? The answer is surprisingly simple.

Bicoloring of Graphs

A bicoloring of a graph is a coloring of each vertex with one of two colors such that no edge is incident to the same color at both ends. Often this is not possible. For example, no graph with a self-loop can be colored in this way. If a bicoloring does exist, the graph is said to be bipartite. In other words, the action of the edges has divided the set of vertices into two partitions. The number of ways two partitions of a set can be colored using two colors is exactly two, thus every bipartite graph has exactly two bicolorings, and they differ only by a rotation of the colors.

Hypermaps

A hypermap is a map bicolored with the colors black and white[1].

The black vertices are called hypervertices, the white vertices are called hyperedges. Given any ordinary map, we can draw it as a hypermap by subdividing each edge with a white vertex: an ordinary map is a hypermap with 2-valent hyperedges.

A hypermap face (a *hyperface*) is bounded by facial walk that alternates between black and white vertices, and thus contains an even number of *half-edges.* The valence of a hyperface is calculated by totaling the number of half-edges (counting with multiplicity if a half-edge appears twice in the walk) and dividing by two.

It is easy to construct the dual of a hypermap: we simply rotate the two colors.

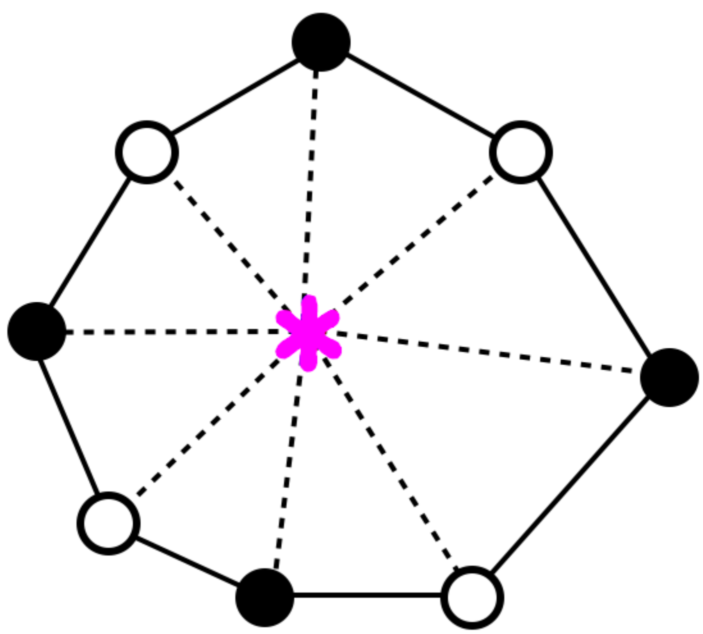

Figure 4 : *The canonical triangulation of a hyperface: the pink star is the new vertex, the dashed lines are the new edges.*

[1] It is the combinatorial explosion of maps that allows us to define a superset as if it were a subset: a hypermap with n hypervertices and m hyperedges is found as a bipartite needle in a haystack of ordinary maps having $(n+m)$ vertices and likely many more than m edges.

The Canonical Triangulation

A small increase in generality can be gained by dealing with hypermaps in dual-pairs rather than individually. From the construction of the hypermap dual we can see that such a generalization coincides with the (uncolored) bipartite maps.

An additional increase in generality is gained by dealing with triples of dual-pairs (what we will call a *six-pack* of hypermaps) all at the same time. A construction called the *canonical triangulation* enables this. Here is the construction: in a hypermap, place a new vertex colored with a third color (we'll use a pink star) in the center of every hyperface, and construct lines to all the vertices—both black and white—in the facial walk. (We do this with multiplicity if a given vertex appears in the facial walk more than once.) The result of this construction is a tricolored, tripartite, triangle-faced map. Every edge connects two vertices of differing color: new lines connect pink to black or pink to white, old lines connect black to white. At every vertex there are an even number of edges (and thus also an even number of triangles) meeting, so the the canonical triangulation is an *Eulerian triangulation.*

The number of ways three partitions of a set can be colored using three colors is exactly six. Trying out each of these six color permutations on a canonical triangulation reveals six different hypermaps (in three dual pairs) each represented by its canonical triangulation. A generalization of hypermaps as six-packs thus coincides with the (uncolored) tripartite triangle-faced maps.

Obtaining Tricolored, Quad-Faced Maps from the Canonical Construction

A tricoloring of the vertices of a triangle induces a 3-coloring of the edges: one may simply mix the two different vertex colors the edge encounters at each end. In our case the three edge colors are gray, dark pink, and light pink. Each triangle face has one edge of each color. On an orientable surface we may speak consistently of counterclockwise (+) or clockwise (−) orientations. On such a surface each triangle can be labelled $+$ or $-$ according to the rotation needed to put the edge colors into gray/dark pink/light pink order. On such a surface, every edge in the triangulation is a $+/-$ boundary. Deleting all edges of a certain color unites neighboring $+$ and $-$ triangles to form a quadrilateral face. The map of course remains tricolored. Every quad has two diagonally opposite vertices that bear the third color, i.e., the color that was not an ingredient in the color of the removed edges. In this way, every six-pack of hypermaps gives us three ways to make a tripartite quad-faced map on the same surface. This will prove handy shortly.

Plain-Woven Baskets

Baskets that are woven in a strictly over-one-under-one pattern (what we call *plain weaving* [2]) are closely related to *alternating links,* the multi-component knots that possess a planar diagram showing the same over-one-under-one pattern. The main differences between alternating links and plain-woven baskets are that baskets not manipulated (that is, we do not attempt to untie them, or alter their configuration) and plain-woven baskets have the option to demonstrate their over-one-under-one pattern on surfaces other than the plane.

In 1876 P.G. Tait [3] showed how an alternating knot could be represented by a checkerboard diagram on the plane. The same sort of diagram can be extended to plain-woven baskets simply by drawing it on a closed surface.[2] Some artists [4-6] have advocated this as a practical approach. In our terms such a diagram

[2]It is a startling fact that all plain-woven baskets are closed: their selvedged borders have exactly the same weave structure as every other weave opening. The brim of a hat is just a weave opening big enough for us to pop our heads through!

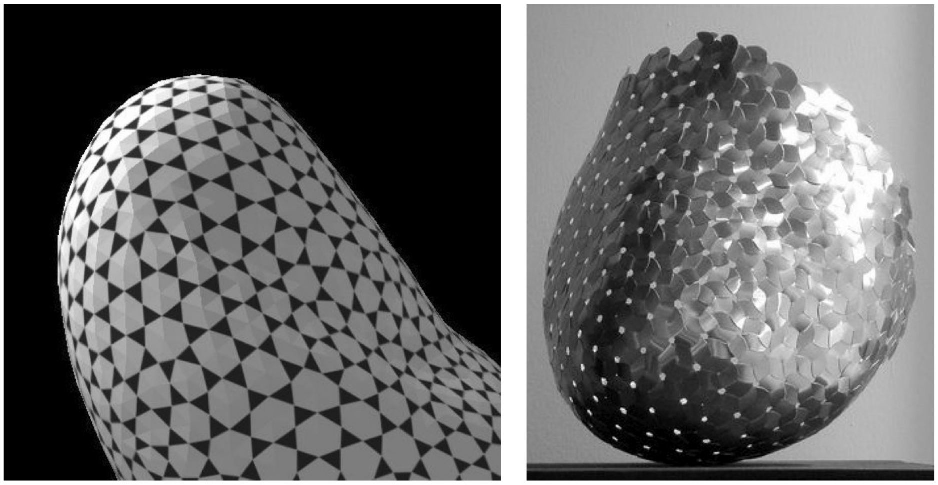

Figure 5 : *A checkerboard diagram of a weave with right-handed triangular openings and the finished weaving.*

is a face 2-colored (a.k.a., *chess-colored*) 4-regular map (every vertex is incident to four edges when counted with multiplicity.) Conventionally, the two colors for the chess coloring are black and white.

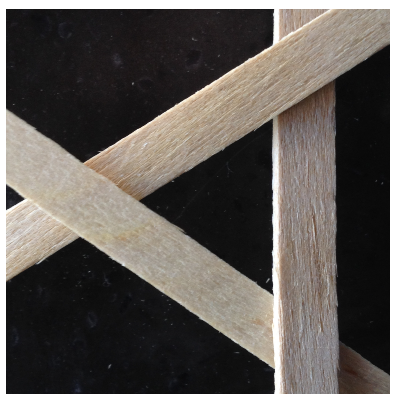

Figure 6 : *Impossible Staircase Handedness (ISH.) One way to assign a handedness to a weave opening is to view it as an impossible staircase. Aligning the curled fingers with the direction of rotation and the extended thumb with the direction of apparent vertical progress, shows that this triangular opening is right-handed. By the ISH convention, a right-handed opening is mapped to the color black.*

A chess-colored 4-regular map explicitly establishes both the path of the weavers and their over-or-under crossings provided we possess a convention to determine the left- or right-handedness of a weave opening and a mapping to the colors black and white. See Figure 6 for one such convention.

Plain-Woven Baskets as Hypermaps

The dual of a chess-colored, 4-regular map is a bicolored quad-faced map. Since every bicolored map is a hypermap, we can represent every plain-woven basket as a hypermap. Further, from every six-pack of hypermaps, we know how to generate three distinct bicolored quad-faced maps. From these, three distinct baskets which can each be woven in both a left-handed weave and a right-handed weave, giving us a total of six baskets from six hypermaps. A suitable convention relating color and handedness (one is advocated in Figure 6) can re-wire this six-on-six relationship into a bijection between hypermaps and plain-woven baskets.

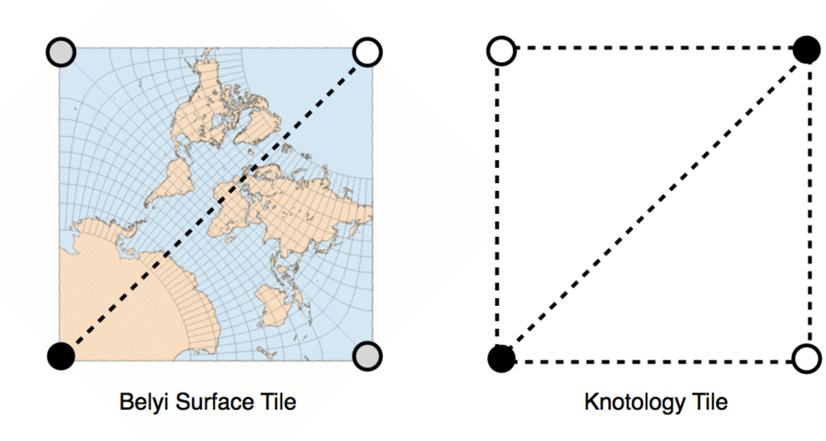

Figure 7 : *Adams's World in a Square II conformal projection of the earth formed into a periodic strip; then folded along its seams and the equator to form a knotology weaver.*

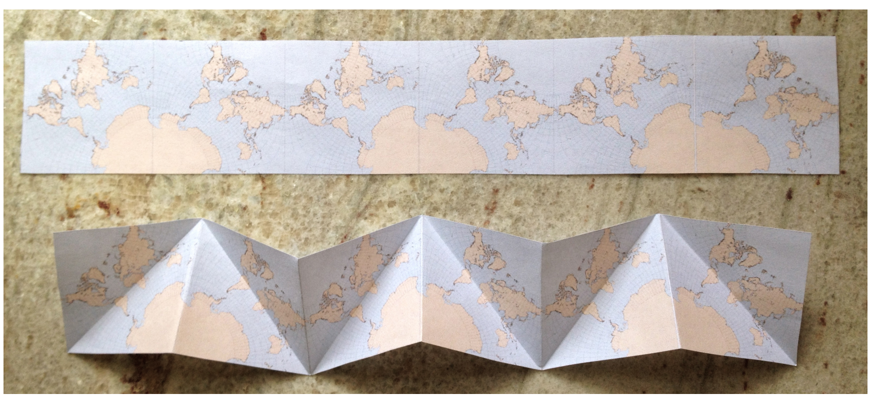

Figure 8 : *Adams's World in a Square II conformal projection of the earth formed into a periodic strip; then folded along its seams and the equator to form a knotology weaver.*

An Example: Ramified Coverings of the Sphere

Identifying plain-woven baskets with hypermaps suggests many interesting math connections. A topic closely related to hypermaps are ramified coverings of the sphere—what the author and other laypeople may prefer to think of as seamless coverings of other surfaces with copes of the earth's surface.

In the terminology of this field, topographic details on the earth's surface would be considered *images,* and their copies on the other surface would be considered *pre-images.* The scheme only works for orientable surfaces, and it is not possible to do the mapping perfectly: certain points on the earth (called *critical values*) must be declared off-limits to visitors and likewise their pre-images on the other surface (called *critical points.*) In general, a minimum of three critical values are needed. A surface that maps to the sphere with at most three critical values is called a *Belyi surface.*

The Adam's World in a Square II projection is a conformal projection of the earth with just three cirtical values: the North and South Poles and the point in the mid-Pacific where the equator and 180°longitude meet. This projection makes a square module or *tile* that is less symmetrical, but quite similar to the tile used in knotology weaving [7]. The Adams projection tile (Figure 7) has vertices in three different colors that must be matched, while the knotology tile has vertices in just two colors that must be matched. In doing the weaving in Figure 1, I found that the North and South Pole critical points could be used at non-Eulerian vertices, namely at the *cube corners* which are frequent motif in knotology.

Conclusion

Plain-woven baskets are closely related to hypermaps. In fact, given suitable conventions of color and handedness, they are in bijection. The many mathematical appearances of hypermaps augur a bright future for basket weaving.

References

[1] Lando, S. K., Zvonkin, A. K., and Zagier, D. B., *Graphs on Surfaces and Their Applications,* Encyclopedia of Mathematical Science, Low Dimensional Topology, Springer-Verlag, 2004.
[2] Akleman, E., et al., "Cyclic plain-weaving on polygonal mesh surfaces with extended graph rotation systems," Proc. SIGGRAPH, 2009.
[3] Tait, P. G., "Some elementary properties of closed plane curves." Messenger of Mathematics, New Series, No.69, 1877, (communicated at the 1876 Meeting of the British Association.)
[4] Snelson, K., "The Nature of Structure: Tensegrity Weaving and the Binary World," www.kennethsnelson.net.
[5] Roelofs, R., "About weaving and helical holes," Bridges 2010.
[6] Mallos, J., "Extra Ways to See: An Artist's Guide to Map Operations," Hyperseeing, Summer 2011.
[7] Strobl, H., "Knotology," www.knotology.eu.